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Some aspects of deformations of supersymmetric field theories

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Abstract: We investigate some aspects of Moyal-Weyl deformations of superspace and their compatibility with supersymmetry. For the simplest case, when only bosonic coordinates are deformed, we consider a four-dimensional supersymmetric field theory which is the deformation of the Wess-Zumino renormalizable theory of a chiral superfield. We then consider the deformation of a free theory of an abelian vector multiplet, which is a non-commutative version of the rank 1 Yang-Mills theory. We finally give the supersymmetric version of the $\alpha' \mapsto 0$ limit of the Born-Infeld action with a B-field turned on, which is believed to be related to the non-commutative U(1) gauge theory.

Keywords: Brane Dynamics in Gauge Theories, Extended Supersymmetry, Superspaces.

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Contents

1. Introduction

Deformations of mathematical structures have been used at different moments in physics. When Galilean transformations between inertial systems were seen not to describe adequately the physical world, a deformation of the group law arose as the solution to this paradox. The Lorenz group is a deformation of the Galilei group in terms of the parameter $1/c$. From the mathematical point of view it is not difficult to imagine the deformation of a group inside the category of groups, but form the physical point of view it has enormous consequences. In this deformation scheme, the old structure is seen as a limit or contraction when the parameter takes a preferred value.

The mathematical structure of quantum mechanics has also an ingredient of deformation with respect to classical mechanics. The first star product or formal deformation of the commutative algebra of classical observables was written in ref. [1, 2]. The star product is a product in the space of formal series in $\hbar$ whose coefficients are functions on the phase space. It is homomorphic to the product of operators in quantum mechanics. In this case the deformation occurs inside the category of algebras, although giving up the commutativity. More complicated features arise in the mathematical framework of quantum mechanics, and the first thing one can realize is that this deformation in the parameter $\hbar$ is a formal deformation (that is, the series in $\hbar$ of a product of two functions is not convergent), unless strong restrictions are made on the functions. Nevertheless, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer realized in their seminal papers [3] that a first approach to a quantum system could be studying the formal deformations of the classical one, leaving aside problems of convergence and of construction of the Hilbert space. The existence and uniqueness of this deformation (up to gauge transformations), finally showed in [4] for any Poisson manifold, supports the belief that the formal deformation encloses the essential information of the quantum system.

Much more recently, non-commutative geometry has entered in physics in different contexts. One context is string theory and M-theory. In their pioneering paper, Connes, Douglas and Schwarz [5] introduced non-commutative spaces (tori) as possible compactification manifolds of space time. Non-commutative geometry arises as a possible scenario for short-distance behavior of physical theories. Since non-commutative geometry generalises standard geometry in using a non-commutative algebra of "functions", it is naturally related to the simpler context of deformation theory.

Matrix theory [6, 7] on non-commutative tori is related to eleven-dimensional supergravity toroidal compactifications with 3-form backgrounds. In this framework, T-duality arises as Morita equivalence in non-commutative geometry [8]. This lead to subsequent developments of Yang-Mills theories on non-commutative tori [9], as the study of their BPS states [10] and a reformulation of T and U dualities of Born-Infeld actions on non-commutative tori [11]. Non-commutative geometry also appeared in the framework of open string theory [13].

Gauge theories on non-commutative spaces had been introduced by Connes [14]. More recently, Seiberg and Witten [15] identified limits in which the entire string dynamics, in presence of a B-field, is described by a deformed gauge theory in terms of a Weyl-Moyal star product on space time. In particular, they showed that the pure quadratic gauge theory with deformed abelian gauge symmetry is related through a change of variables to a non-polynomial gauge theory with undeformed gauge group (see also [16]). This brings a connection between the Born-Infeld action and gauge theories in non-commutative spaces. In view of the fact that the supersymmetric Born-Infeld [18] action naturally arises as the Goldstone action of $N = 2$ supersymmetry, partially broken to $N = 1$ [19], it must be the case that this interpretation should have its counterpart in the framework of deformed gauge theory. This connection will be clarified in this paper. Subsequently, this deformation of space time was used for ordinary four-dimensional field theories with a B-field of maximal rank in $\mathbb{R}^4$ space time. It was shown that the deformed theories enjoy unsuspected renormalization properties as well as UV/IR connection reminiscent of string theory [17].

Other approaches connecting deformation theory to theories of gravity have also appeared in the literature. Among others, we can mention the deformation quanti-

JHEP05(2000)008

zation of M-theory [22], quantum anti-de Sitter space time [23], q-gravity [24] and gauge theories of quantum groups [25].

In this paper some issues related to theories formulated in deformations of superspace are investigated. Aspects of non-commutative supergeometry [21, 20] and non-commutative supersymmetric field theories [41, 42, 43] were recently considered in the literature. Section 2 is a self-explanatory account of the Moyal-Weyl deformation generalised to superspace. In particular we show that the deformation of a Grassmann algebra obtained with a Weyl ordering rule is a Clifford algebra once we specify a value for the parameter (non-formal deformation). In section 3 we consider the compatibility of this deformation with the supertranslation group and we write a new star product which is covariant under the action of such supergroup. In section 4 we present the simplest example of a deformed supersymmetric field theory, the Wess-Zumino model and we give its explicit expression in terms of field components. In section 5 we consider deformed gauge groups in superspace and derive a non-commutative version of rank 1 Yang-Mills theory which is then coupled to chiral superfields. This novel construction allows us to generalize non-commutative supersymmetric field theories to extended supersymmetry, as well as to gauge groups of arbitrary rank. We show that rank 1 N-extended super-Yang-Mills theories are invariant under shifts of the N gauginos by N constant, anti-commuting parameters, so they can be regarded as Goldstone actions of $2N$ supersymmetries spontaneously broken to N supersymmetries. We also prove that to first order in the deformation parameter, the change of variables of Seiberg and Witten [15] to convert the rank 1 non-commutative quadratic gauge theory to a commutative higher derivative theory is consistent with supertranslations.

Finally in section 6 we derive the $\alpha' \mapsto 0$ limit of supersymmetric Born-Infeld actions, which are the starting point for comparisons with super-Yang-Mills theories in non-commutative superspaces.

2. Star product in superspace

2.1 Super Poisson bracket

A super-vector space over $\mathbb{R}$ or $\mathbb{C}$ is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ with grading $p = 0, 1$. On V we can give an associative operation $\cdot : V \otimes V \mapsto V$ respecting the grading, that is

$$p(a \cdot b) = p(a) + p(b)\,.$$

Then V is a superalgebra. A super-Lie algebra is a super-vector space V with a bracket $[\ ,\ ] : V \otimes V \mapsto V$ satisfying

$$[X, Y] = -(-1)^{p_X p_Y}[Y, X]\,, \tag{2.1}$$

$$[X, [Y, Z]] + (-1)^{p_Z(p_X+p_Y)}[Z, [X, Y]] + (-1)^{p_X(p_Y+p_Z)}[Y, [Z, X]]\,. \tag{2.2}$$

The superspace of dimension (p,q) is the affine space $\mathbb{R}^p$ together with a superalgebra $\mathcal{S}^{p,q} = C^\infty(\mathbb{R}^p)\otimes\Lambda(\mathbb{R}^q)$ (the algebra of "functions" on superspace), where $\Lambda(\mathbb{R}^q) = \sum_{i=0}^q \Lambda^i(\mathbb{R}^q)$ is the exterior algebra of q symbols $\theta^1, \theta^2, \dots, \theta^q$. We assign grade one to the symbols θ^i. It is clear what are the even and odd subspaces.

An element of this superalgebra can be written as

$$a(x,\theta) = a_0(x) + a_i(x)\theta^i + a_{i_1 i_2}\theta^{i_1}\wedge\theta^{i_2} + \cdots + a_{i_1 i_2 \dots i_q}\theta^{i_1}\wedge\theta^{i_2}\cdots\wedge\theta^{i_q}\,,$$

where $a_{i_1 i_2 \dots i_j}$ is antisymmetric in all its indices. It is a commutative superalgebra, that is,

$$a\cdot b = (-1)^{p_a p_b} b\cdot a$$

for homogeneous elements of degrees p_a and p_b.

A left derivation of degree $m = 0,1$ of a superalgebra is a linear map $\partial^L : V \mapsto V$ such that

$$\partial^L(a\cdot b) = \partial^L(a)\cdot b + (-1)^{m p_a} a\cdot\partial^L(b)\,.$$

Graded left derivations form a $\mathbb{Z}_2$-graded vector space. Any linear map L can be decomposed as the sum $L = L_0 + L_1$, where L_0 and L_1 are maps of degree 0 (they preserve the degree) and 1 (they change the degree) respectively. If the superalgebra is commutative, an even derivation has degree 0 as a linear map and an odd derivation has degree 1 as a linear map.

In the same way right derivations are defined,

$$\partial^R(a\cdot b) = (-1)^{m p_b}\partial^R(a)\cdot b + a\cdot\partial^R(b)\,.$$

Notice that derivations of degree zero are both, right and left.

A super Poisson structure on a commutative (this condition could be relaxed, in particular, to introduce matrix valued superfields) superalgebra is a super-Lie algebra structure $\{\ ,\ \}$ on it which is also a bi-derivation with respect to the commutative superalgebra product. More specifically, it satisfies the following derivation property on homogeneous elements,

$$\{a, b\cdot c\} = \{a,b\}\cdot c + (-1)^{p_a p_b} a\cdot\{b,c\}\,,$$

which together with the anti-symmetry property (2.1) implies

$$\{b\cdot c, a\} = b\cdot\{c,a\} + (-1)^{p_a p_c}\{b,a\}\cdot c\,.$$

So, for example, if a is even, $\{a,\cdot\}$ is a derivation of degree zero, and if it is odd it is a left derivation of degree 1.

Example. Consider the superalgebra $\mathcal{S}^{p,2}$, with elements

$$\Phi(x,\theta) = \Phi_0(x) + \Phi_\alpha(x)\theta_\alpha + \Phi_{\alpha\beta}(x)\theta_\alpha \wedge \theta_\beta \,.$$

The derivations ∂_i defined as

$$\partial_i\Phi(x,\theta) = \partial_i\Phi_0(x) + \partial_i\Phi_\alpha(x)\theta_\alpha + \partial_i\Phi_{\alpha\beta}(x)\theta_\alpha \wedge \theta_\beta$$

are even derivations. The left derivations ∂^L_α defined as

$$\partial^L_\alpha\Phi(x,\theta) = \Phi_\alpha(x) + 2\Phi_{\alpha\beta}(x)\theta_\beta$$

are odd. We can also define right derivations,

$$\partial^R_\alpha\Phi(x,\theta) = \Phi_\alpha(x) + 2\Phi_{\beta\alpha}(x)\theta_\beta \,.$$

Notice that $\partial^R_\alpha = \partial^L_\alpha$ on odd elements and $\partial^R_\alpha = -\partial^L_\alpha$ on even elements. This implies that $[\partial^R_\alpha, \partial^L_\beta]_- = 0$.

These definitions can easily be extended to algebras with bigger odd dimension in an obvious manner.

As an example, consider the following super Poisson bracket

$$\{\Phi,\Psi\} = P^{ab}\partial_a\Phi\,\partial_b\Psi + P^{\alpha\beta}\partial^R_\alpha\Phi\,\partial^L_\beta\Psi = P^{AB}\partial^R_A\Phi\,\partial^L_B\Psi \,. \tag{2.3}$$

where P is a constant matrix and satisfies the symmetry properties

$$P^{ab} = -P^{ba}\,, \qquad P^{\alpha\beta} = P^{\beta\alpha}\,.$$

It is easy to see that it satisfies the Jacobi identity (2.2).

2.2 Super-star product

A generalisation of the Moyal-Weyl [1, 2] deformation of $C^\infty(\mathbb{R}^n)$ to $\mathcal{S}^{p,q}$ exists. This algebra structure corresponds to the quantization of systems with both, bosonic and fermionic degrees of freedom, and it was studied by Berezin as early as in [26]. There, the quantization was studied in terms of products of Weyl symbols of operators, very much in the same spirit than [1, 2]. In a language closer to ours, it appeared in [27] and in [28, 29].

We remind that a deformation of the commutative algebra $C^\infty(\mathbb{R}^p)$ is an associative product on the space of formal series on a parameter h with coefficients in $C^\infty(\mathbb{R}^n)$, that is $C^\infty(\mathbb{R}^n)[[h]] = \mathbb{R}[[h]] \otimes C^\infty(\mathbb{R}^n)$. The term of first order in h, antisymmetrized, is always a Poisson bracket.

We denote by $P(f \otimes g) = \{f, g\}$ a super Poisson bracket like (2.3), in a space of arbitrary odd dimension. A deformation of the commutative superalgebra $\mathcal{S}^{p,q}$ is then given by

$$\begin{aligned} \star : \mathcal{S}^{p,q}[[h]] \otimes \mathcal{S}^{p,q}[[h]] &\longrightarrow \mathcal{S}^{p,q}[[h]], \\ f \otimes g &\longmapsto e^{hP}(f \otimes g)\,, \end{aligned} \tag{2.4}$$

where we have denoted

$$e^{hP} = \sum_{n=0}^{\infty} \frac{h^n}{n!} P^n$$

with

$$P^n(f \otimes g) = P^{A_1B_1}P^{A_2B_2}\cdots P^{A_nB_n}(\partial^R_{A_1}\partial^R_{A_2}\cdots\partial^R_{A_n})f \cdot (\partial^L_{B_1}\partial^L_{B_2}\cdots\partial^L_{B_n}g)\,.$$

(We remind here that P^{AB} is a constant matrix.) For $q = 0$ this is the standard Moyal-Weyl deformation. Notice that the first order term is exactly the super Poisson bracket. The proof of the associativity of this product is parallel to the one developed in [3] for the bosonic case.

2.3 Non-formal deformation

We consider the associative algebra over $\mathbb{R}[[h]]$, $\mathcal{A}^{p,q}$, generated by the symbols $X^1, \ldots, X^p$, $\Theta^1, \ldots, \Theta^q$ and the relations given by the super Poisson bracket (2.3).

$$[X^a, X^b]_- = hP^{ab}\,, \tag{2.5}$$

$$[\Theta^\alpha, \Theta^\beta]_+ = hP^{\alpha\beta}\,, \tag{2.6}$$

where h is a formal parameter. Since X's and Θ's commute, it is clear that $\mathcal{A}^{p,q} \approx U_h^p \otimes \Lambda_h^q$, where U_h^p is the associative algebra over $\mathbb{R}[[h]]$ generated by the symbols X's and relations (2.5) and Λ_h^q is the associative algebra over $\mathbb{R}[[h]]$ generated by the symbols Θ's and relations (2.6). $\mathcal{A}^{p,q}$ is isomorphic to $(\mathrm{Pol}(\mathbb{R}^p) \otimes \Lambda(\mathbb{R}^q)[[h]], \star)$, (polynomials are closed under the $\star$ operation). To prove this, we take a basis in $\mathrm{Pol}(\mathbb{R}^p)[[h]]$,

$$x^{i_1} \cdot x^{i_2} \cdots x^{i_n}, \qquad i_1 \leq i_2 \leq \cdots \leq i_n\,. \tag{2.7}$$

There is an $\mathbb{R}[[h]]$-module isomorphism $\mathrm{Sym} : \mathrm{Pol}(\mathbb{R}^p)[[h]] \mapsto U_h^p$ mapping the elements of the basis (2.7) in the following way:

$$\begin{aligned}\mathrm{Sym}\,(x^{i_1}x^{i_2}\cdots x^{i_n}) &= \frac{1}{n}\sum_{\sigma\in S_n} X^{\sigma(i_1)} \cdot X^{\sigma(i_2)} \cdots X^{\sigma(i_n)} \\ &= \exp(X^i\partial_i)(x^{i_1}x^{i_2}\cdots x^{i_n})|_{x^{i_k}=0}\,,\end{aligned}$$

which is the usual Weyl or symmetric ordering. The product in $\mathrm{Pol}(\mathbb{R}^p)[[h]]$ defined by

$$\mathrm{Sym}^{-1}(\mathrm{Sym}(f) \cdot \mathrm{Sym}(g)) \tag{2.8}$$

is equal to $\star$ in (2.4) restricted to polynomials. A proof of this fact is given in [2] (where indeed, the argument is extended to C^∞ functions).

Consider the basis in $\Lambda(\mathbb{R}^q)[[h]]$

$$\theta_{i_1} \wedge \theta_{i_2} \wedge \cdots \wedge \theta_{i_n}\,, \quad i_1 \leq i_2 \leq \cdots \leq i_n\,. \tag{2.9}$$

$((\dim \Lambda(\mathbb{R}^q))=2^q)$. We define the isomorphism $\mathrm{Sym} : \Lambda(\mathbb{R}^q)[[h]] \mapsto \Lambda_h^q$ as

$$\mathrm{Sym}\,(\theta_{i_1} \wedge \theta_{i_2} \wedge \cdots \wedge \theta_{i_n}) = \Theta_{i_1}\Theta_{i_2}\cdots\Theta_{i_n}$$

for the elements of the basis (2.9). This is the equivalent of the Weyl ordering for odd generators. One can see directly by inspection that the product defined on $\Lambda(\mathbb{R}^q)[[h]]$ by this isomorphism is the same than $\star$ in (2.4) restricted to the exterior algebra. So we can conclude that the algebra generated by X, Θ and relations (2.5) and (2.6) is isomorphic to the $\star$-product algebra. Given any $\mathbb{R}[[h]]$-module isomorphism among $\mathrm{Pol}(\mathbb{R}^p)[[h]]$ and U_h^p, one can construct a star product as in (2.8). The resulting (isomorphic) star products are called equivalent.

For polynomials, the formal parameter h can be specialized to any real value and one obtains a convergent star product. We want to look closer to this algebra over $\mathbb{R}$.

By a linear change of coordinates $P \mapsto A^T P A$, we can always bring the matrices P^{ab} and $P^{\alpha\beta}$ into a canonical form [30], that is

$$P^{ab} = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P^{\alpha\beta} = \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \eta_q \end{pmatrix}, \tag{2.10}$$

where $\eta_\alpha = \pm 1$ for $\alpha = 1, \ldots, q'$ and $\eta_\alpha = 0$ for $\alpha = q'+1, \ldots, q$.

Denote $q' = m + n$, where $\eta_\alpha = -1$ for $\alpha = 1, \ldots, m$ and $\eta_\alpha = +1$ for $\alpha = m+1, \ldots, m+n$. It is obvious that Λ_h^q, evaluated for a real value of h is isomorphic to the Clifford algebra $\mathcal{C}(m,n)$ tensor product with the exterior algebra on the remaining $q - q'$ generators, which does not get deformed. (The isomorphism is given by $\gamma_\alpha = \sqrt{2h}\Theta_\alpha$.) This relation with Clifford algebras was noticed in [26].

If $m = n$, we can make again a linear change of variables that brings $P^{\alpha\beta}$ to the form

$$P^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We have then n pairs of canonically conjugate fermionic variables. The Poisson bracket for fermionic variables was first written in [31].

3. Formal deformations of rigid supersymmetry

We are interested in describing physical theories defined on a deformation of superspace. Superfields are used as basic objects of such theories. Mathematically, they are a generalisation of the superalgebra $\mathcal{S}^{p,q}$. Consider the trivial bundle over $\mathbb{R}^p$ with fibre the Grassmann or exterior algebra $\Lambda(\epsilon_1, \ldots, \epsilon_n)$. Consider the algebra of sections on that bundle, $\Gamma^q(\mathbb{R}^p)$ and the tensor product $\Phi_n^{p,q} = \Gamma^n(\mathbb{R}^p) \otimes \Lambda^n$.

$\Phi_n^{p,q}$ is a commutative superalgebra with the product defined as usual

$$(a \otimes \Psi_1)(b \otimes \Psi_2) = (-1)^{p_1 p_b} ab \otimes \Psi_1 \Psi_2 \,, \qquad a, b \in \Gamma^n(\mathbb{R}^p)\,, \quad f, g \in \Lambda^n \,,$$

and the left and right $\Gamma^n(\mathbb{R}^p)$-module structures are given by

$$b(a \otimes \Psi) = (ba \otimes \Psi), \qquad (a \otimes \Psi)b = (-1)^{p_b p_\Psi}(ab \otimes \Psi)\,.$$

The rank (n) of the trivial bundle can be chosen $n = \dim \Lambda^q = 2^q$. Then scalar superfields are an even subalgebra of $\Phi_n^{p,q}$, generated by elements of the form

$$\Phi(x, \theta) = \Phi_0(x) + \theta_i \otimes \Phi_i(x) + \theta_j \wedge \theta_j \otimes \Phi_{ij} + \cdots \,,$$

where $\Phi_{i_1 i_2 \ldots i_k}$ are independent global sections in $\Gamma^q(\mathbb{R}^p)$, antisymmetric in the indices $i_1 i_2 \ldots i_k$.

One can extend (2.3) and (2.4) to $\Phi_n^{p,q} \otimes \mathbb{R}[[h]]$ by linearity. It follows that $(\Phi_n^{p,q}[[h]], \star)$ is a non-commutative superalgebra.

In what follows, we will restrict ourselves to four-dimensional space time, although all considerations could be easily extended to other dimensions. We consider the superspace associated with the four-dimensional N-extended Poincaré supersymmetry, with coordinates (or generators) $\{x^\mu, \theta^{\alpha i}, \bar{\theta}_i^{\dot{\alpha}}\}$, where $\{x^\mu\}$ are the coordinates of ordinary four-dimensional Minkowski space $\mathcal{M}$, and $\{\theta^{\alpha i}, \bar{\theta}_i^{\dot{\alpha}}\}$ are Weyl spinors under the Lorentz group. We are interested in deformations of this superspace such that they have an action of the supertranslation group. The odd supertranslations with parameters $\epsilon^{\alpha i}, \bar{\epsilon}_i^{\dot{\alpha}}$, act on the generators of superspace as

$$\begin{aligned} x^\mu &\longmapsto x'^\mu = x^\mu + i(\theta^{\alpha i}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}_i^{\dot{\alpha}} - \epsilon^{\alpha i}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}_i^{\dot{\alpha}})\,, \\ \theta^{\alpha i} &\longmapsto \theta'^{\alpha i} = \theta^{\alpha i} + \epsilon^{\alpha i}\,, \\ \bar{\theta}_i^{\dot{\alpha}} &\longmapsto \bar{\theta}'^{\dot{\alpha}}_i = \bar{\theta}_i^{\dot{\alpha}} + \bar{\epsilon}_i^{\dot{\alpha}}\,. \end{aligned} \tag{3.1}$$

By convention, we write a scalar superfield as

$$\Phi(x^\mu, \theta^{\alpha i}, \bar{\theta}_i^{\dot{\alpha}}) = \Phi_0(x) + \theta^{\alpha i}\Psi_\alpha^i + \bar{\theta}_i^{\dot{\alpha}}\bar{\Sigma}_{\dot{\alpha} i} + \theta^{\alpha i}\theta^{\beta j}\Psi_{\alpha\beta}^{ij} + \cdots \,,$$

where we have dropped the symbols "$\wedge$" and "$\otimes$". Let $g(\epsilon)$ be a super translation as in (3.1). The action of g on superfields is given by $(g^{-1}\Phi)(x, \theta, \bar{\theta}) = \Phi(x', \theta', \bar{\theta}')$. We require that the super translation group acts as a group of automorphisms on the deformed algebra, that is

$$g(\Phi_1 \star \Phi_2) = (g\Phi_1) \star (g\Phi_2)\,.$$

It is convenient to introduce the right and left odd derivations called super-covariant derivatives,

$$D^{R,L}{}_{\alpha i} = \partial^{R,L}{}_{\alpha i} + (i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_i\partial_\mu)^{R,L},$$
$$\bar{D}^{R,Li}_{\dot{\alpha}} = -\bar{\partial}^{R,Li}_{\dot{\alpha}} - (i\theta^{\alpha i}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_\mu)^{R,L}\,.$$

They have the property that

$$D^{R,L}{}_{\alpha i}(g\Phi) = g(D^{R,L}{}_{\alpha i}\Phi)$$

and the same for $\bar{D}^{R,Li}_{\dot{\alpha}}$. We define a Poisson bracket

$$\{\Phi,\Psi\} = P^{\mu\nu}\partial_\mu\Phi\partial_\nu\Psi + P^{\alpha i\beta j}D^{R}{}_{\alpha i}\Phi D^{L}{}_{\beta j}\Psi\,. \tag{3.2}$$

The crucial properties are that $[D^{R}{}_{\alpha i}, D^{L}{}_{\beta j}]_- = 0$ and that $D^{R,L}_{\alpha i}P^{AB} = 0$ ($A = \mu, \{\alpha\, i\}$), so one can again construct a Weyl-Moyal star product as in (2.4), which will also be covariant under the supertranslation group. One could also extend (3.2) by using $N-k$ D's and k $\bar{D}$'s, $k = 1,\ldots,N$, which can be taken to anticommute. Notice that this Poisson structure is degenerate in the space of odd variables.

A chiral field [38] satisfies the constraint $\bar{D}^{R,Li}_{\dot{\alpha}}\Phi = 0$. The solution of this equation can be written (after a change of variables) as $\Phi(x,\theta)$. Chiral superfields are a subalgebra under ordinary multiplication, but they are not closed under the star product defined with (3.2) unless $P^{\alpha i\beta j} = 0$.

When considering extended supersymmetry this notion of chirality can be generalised. The R-symmetry group $\mathrm{U}(N)$ acts by automorphisms on the super-Poincaré algebra, leaving invariant the even generators. For $N > 1$, one can take the direct product of the Minkowski space with the flag manifold $\mathrm{SU}(N)/\,\mathrm{U}(1)^{N-1}$, and consider a supermanifold structure of odd dimension $4N$ on it. This is constructed by taking the quotient $(\mathcal{L}\otimes\mathrm{SU}(N))\otimes_s\mathcal{ST}/\mathcal{L}\otimes\mathrm{U}(1)^{N-1}$, where $\mathcal{L}$ is the Lorentz group and $\mathcal{ST}$ is the supertranslation group. It it is called harmonic superspace [32]. The algebra of global sections (or functions) on the resulting supermanifold is isomorphic to

$$C^\infty(\mathcal{M}\times\mathrm{SU}(N)/\,\mathrm{U}(1)^{N-1})\otimes\Lambda^{4N}\,. \tag{3.3}$$

This isomorphism is not canonical, since it is not preserved by supersymmetry transformations, but it is preserved by the action of the R-symmetry group.

Let us denote the coordinates in an open set as $\{x^\mu, u, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha}}_i\}$ where u is a unitary matrix or coset representative of $\mathrm{SU}(N)/\,\mathrm{U}(1)^{N-1}$. With the coset representatives one can define rotated covariant derivatives

$$\mathcal{D}_{\alpha I} = u^i_I D_{\alpha i}, \qquad \bar{\mathcal{D}}^I_{\dot{\alpha}} = u^i_I\bar{D}^i_{\dot{\alpha}}\,.$$

The advantage of such formulation is that the notion of chiral superfield can be generalised by imposing the following R-symmetry covariant constraints on the superfields $\Phi(x,u,\theta,\bar{\theta})$,

$$\mathcal{D}_{\alpha 1}\Phi = \cdots = \mathcal{D}_{\alpha k}\Phi = 0 = \bar{\mathcal{D}}^{k+1}_{\dot{\alpha}}\Phi = \cdots = \bar{\mathcal{D}}^N_{\alpha}\Phi\,.$$

(No superscript will mean that we are considering a left derivative.) The solution of these constraints can be expressed as superfields that do not depend on k θ's and $N-k$ $\bar{\theta}$'s ($k=0,N$ being the usual chiral and anti-chiral superfields). These fields have been called "Grassmann analytic" in the literature [33] and, as chiral superfields, they form a subalgebra.

One can consider deformations of this supermanifold for a given super Poisson structure. In particular one can consider a deformation affecting only the first factor in (3.3). Any deformation of this form will have the supersymmetry algebra as an algebra of derivations. As the simplest case, let us take a non-trivial Poisson bracket only in the directions of $\mathcal{M}$,

$$iP = iP^{\mu\nu}\frac{\partial}{\partial x^{\mu}} \otimes \frac{\partial}{\partial x^{\nu}}$$

that is, the Poisson bracket of two (complex) superfields is

$$\{\Phi_1, \Phi_2\} = iP^{\mu\nu}\,\frac{\partial \Phi_1}{\partial x^{\mu}}\,\frac{\partial \Phi_2}{\partial x^{\nu}}\,,$$

where $P^{\mu\nu}$ is an arbitrary constant anti-symmetric matrix and $\Phi_i(x,u,\theta)$ arbitrary superfields. Then, the Weyl-Moyal star product on the algebra of superfields is given by

$$\Phi_1 \star \Phi_2 = \exp(iP)(\Phi_1 \otimes \Phi_2)\,. \tag{3.4}$$

It is clear from this expression that Grassmann analytic superfields are closed under the star product (3.4), as chiral superfields are.

4. Non-commutative Wess-Zumino model

The simplest example of an $N=1$ supersymmetric field theory is the Wess-Zumino model, whose action is

$$\int d^4x d^{\theta} d^2\bar{\theta}\;\Phi\bar{\Phi} + \int d^4x\;\left(\int d^2\theta\;\left(\frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\right) + \text{c.c.}\right),$$

where the chiral superfield Φ has the expansion

$$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,,$$

where $y = x + i\theta\sigma\bar{\theta}$. A formal deformation of this action can be written using the star product defined above (3.4),

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Phi\bar{\Phi} + \int d^4x\;\left(\int d^2\theta\;\left(\frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^{\star 3}\right) + \text{c.c.}\right), \tag{4.1}$$

JHEP05(2000)008

where $\Phi^{\star n} = \Phi \star \Phi \cdots (n) \cdots \star \Phi$. This model was also considered in [41, 42]. This lagrangian is unique (for every star product) as a consequence of the fact that

$$\int d^4x\ A \star B = \int d^4x\ AB = \int d^4x\ B \star A \tag{4.2}$$

and

$$\int d^4x\ A_1 \star \cdots \star A_n = \int d^4x\ A_{\sigma(1)} \star \cdots \star A_{\sigma(n)}\,, \tag{4.3}$$

where σ is a cyclic permutation of $(1, \ldots, n)$. Notice also that the above action is real, since

$$\overline{A \star B} = \bar{B} \star \bar{A}\,.$$

As a consequence of (4.2) and (4.3), the auxiliary field F satisfies pure algebraic equations

$$F = -m\bar{A} - g\bar{A} \star \bar{A}$$

so the component form of the lagrangian is

$$\begin{aligned} & i\partial_\mu \bar{\psi}\bar{\sigma}^\mu \psi + \bar{A}\partial_\mu \partial^\mu A - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi}) - m^2 \bar{A}A - g(A(\psi \star \psi) + \bar{A}(\bar{\psi} \star \bar{\psi})) - \\ & \quad - mg(A(\bar{A} \star \bar{A}) + \bar{A}(A \star A)) - g^2 (A \star A)(\bar{A} \star \bar{A})\,. \end{aligned} \tag{4.4}$$

The non-deformed Wess-Zumino model is a renormalizable field theory which only requires a (logarithmically divergent) wave function renormalization [35, 36]. This is due to supersymmetric non-renormalization theorems of chiral terms [37].

The deformed Wess-Zumino model is the supersymmetric extension of the ϕ^4 theory considered in [17] where the model was proven to be "renormalizable" in some extended sense. Consequently, we expect that its supersymmetric extension is also "renormalizable". Moreover, since the interactions are purely chiral, no quadratic divergences appear and then the UV/IR connection induced by extra poles in the propagator [17] does not appear in this model [41]. Additional aspects of the UV/IR connection in non-commutative supersymmetric models are discussed in [42, 43].

The non-deformed lagrangian has a quartic interaction that is invariant under a local U(1) symmetry. This invariance is inherited from the superconformal symmetry present in the model when $m = 0$. The deformed lagrangian only preserves the global U(1) invariance. It is interesting to observe that there is another possible quartic term

$$(\bar{A} \star A)^2\,, \tag{4.5}$$

which is invariant under a non-commutative local U(1)-symmetry [34]. Supersymmetry picks the first choice without any contradiction because the R-symmetry is not deformed.

Incidentally, it was also shown that the pure bosonic theory of a complex scalar field A with the quartic interaction as given in lagrangian (4.4) is not renormalizable unlike the theory with the quadratic invariant (4.5). This is not a contradiction because in the Wess-Zumino model the additional interactions due to supersymmetry are responsible for the cancellation of dangerous divergences (in particular quadratic divergences).

As a side remark, we note that while the interaction $\bar{A}\star\bar{A}\star A\star A$ is typical of an F-term, the other possibility $\bar{A}\star A\star\bar{A}\star A$ is typical of a D-term, so we expect the latter to occur in the deformed version of supersymmetric QED. Even more, both quartic terms occur and in fact are related one to another when N-extended supersymmetry is present. This will be the case in the deformed version of $N=2,4$ super-Yang-Mills theories, which in addition require a deformation of the gauge symmetry.

5. Non-commutative rank 1 gauge theory in superspace

In this section we introduce a deformation of an abelian gauge theory in superspace [39, 40]. The gauge group is a group of formal series in a parameter with coefficients which are chiral superfields (U, with $\bar{D}_{\dot\alpha}U=0$). The multiplication law is given by the star product in (3.4)

$$U_1\star U_2=U_3\,,$$

which preserves chirality. We will denote by $U^{\star-1}$ the inverse with respect to the star product,

$$U\star U^{\star-1}=U^{\star-1}\star U=1\,.$$

Notice that for $\theta=\bar\theta=0$, the gauge parameter is a complex function, so the gauge group is the complexification of U(1). We can write an element U as

$$U=e^{\star i\Lambda}=\sum_{n=0}^{\infty}\frac{1}{n!}(i\Lambda)^{\star n}\,,$$

and then

$$U^{\star-1}=e^{\star-i\Lambda}\,,\qquad U^{\dagger}=e^{\star-i\bar\Lambda}\,,\qquad U^{\dagger\star-1}=e^{\star i\bar\Lambda}\,.$$

We introduce a connection superfield V [40] which transforms under the gauge group as

$$\begin{aligned}e^{\star V}&\longmapsto U^{\dagger}\star e^{\star V}\star U\\ e^{\star-V}&\longmapsto U^{\star-1}\star e^{\star-V}\star U^{\dagger\star-1}.\end{aligned}$$

The non-commutative field strength

$$W_\alpha=\bar{D}^2\left(e^{\star-V}\star D_\alpha e^{\star V}\right)\,,\qquad \bar{D}_{\dot\alpha}W_\alpha=0\,,$$

transforms as

$$W_\alpha\longmapsto U^{\star-1}\star W_\alpha\star U\,.$$

The action

$$\mathcal{S}_{NCYM} = \int d^4x \, \left(\int d^2\theta \; W_\alpha \star W^\alpha + \text{c.c.} \right) \tag{5.1}$$

defines the non-commutative rank 1 gauge theory. It is gauge invariant as a consequence of (4.2). If we set the gaugino λ and the auxiliary field D to zero, this action reduces to the bosonic non-commutative action considered in [15]. Note also that the equation of motion of the auxiliary field is $D = 0$.

The infinitesimal gauge transformation of the connection superfield is an infinite power series with terms of the type $V^{\star n}$. To first order in V it is

$$\delta V = i(\Lambda - \bar{\Lambda}) - \frac{1}{2} i[(\Lambda + \bar{\Lambda}) \star V - V \star (\Lambda + \bar{\Lambda})] \, .$$

This is actually the transformation in the Wess-Zumino gauge ($V^{\star 3} = 0$) [39, 40]. In this gauge the field strength becomes

$$W_\alpha = D_\alpha V - \frac{1}{2}(V \star D_\alpha V - D_\alpha V \star V) \, .$$

Since the Wess-Zumino gauge depends on the deformation parameter P, the modified supersymmetry transformations which preserve this gauge will also depend on P. Indeed, the gaugino transformation contains the two-form field strength $F = dA + iA \star A$. Also, the supersymmetry transformation of the auxiliary field D contains the covariant derivative of the gaugino $\nabla\lambda = d\lambda + i(A \star \lambda - \lambda \star A)$.

It is worth noticing that the action in (5.1) is invariant under a non-linearly realised supersymmetry transformation

$$\delta W_\alpha = \eta_\alpha \, ,$$

where η_α is a constant, anti-commuting spinor. This leads to the interpretation of a non-commutative Yang-Mills theory as a Goldstone action of partial breaking of supersymmetry.

We may now couple this action to matter chiral multiplets S_i. This can be done in two different ways, depending whether we introduce adjoint matter $S \longmapsto U^{\star -1} \star S \star U$ (which is neutral in the commutative limit), or charged matter $S \mapsto S \star U$. In the first case the non-commutative gauge invariant coupling is

$$\int d^4x \int d^2\theta \, d^2\bar{\theta} \; S \star e^{\star - V} \star \bar{S} \star e^{\star V} \, . \tag{5.2}$$

Note that we can add to the action any chiral interaction such as (4.1), which will be automatically gauge invariant.

If we now consider a single chiral multiplet and a vector multiplet, then the sum of the two actions (5.1) and (5.2) is known to have in the commutative limit $N = 2$ supersymmetry. Therefore, following the discussion in section 3, the deformed theory is the first example of deformed theory with $N=2$ supersymmetry. This theory could in fact be reformulated using harmonic superspace which is the natural set up for $N=2$ Yang-mills theories [32].

If we introduce three chiral adjoint multiplets $S_i,\ i=1,\ldots,3$ with an additional self coupling

$$\int d^2\theta\ \epsilon^{ijk} S_i \star S_j \star S_k + \text{c.c.}\,,$$

(which vanishes in the commutative limit) we obtain a deformation of $N=4$, rank 1 supersymmetric Yang-Mills theory, which could also be reformulated in harmonic superspace [44, 45]. The Yang-Mills field is then a Grassmann analytic function [32, 33] and it is important that the star product (3.4) preserves Grassmann analyticity.

N-extended non-commutative rank 1 gauge theories are invariant under N non-linearly realized supersymmetries, corresponding to constant shifts of the gauginos by anti-commuting parameters. This is due to the fact that the cubic interactions involving gauginos, under such transformation, vary into a Moyal bracket (antisymmetrized star product), which is a total space time derivative. This brings more evidence to the fact that such theories are closely connected to world volume brane theories which, as a microscopic description of 1/2 BPS states, have $2N$ supersymmetries, with half of them non-linearly realized in the spontaneously broken phase [19, 48].

For a charged matter field, the gauge invariant non-commutative action is

$$\int d^4x \int d^2\theta\, d^2\bar\theta\ S \star e^{\star -V} \star \bar S\,.$$

It was shown in [15] that the phase space of a non-commutative Yang-Mills theory can be mapped to the phase space of an ordinary Yang-Mills theory by a "change of variables" realised in the following way: the gauge potential of the ordinary gauge group A is mapped into the gauge potential $\hat A(A)$ of the non-commutative (deformed) gauge group , while the gauge group parameter λ is mapped into the non-commutative gauge group parameter $\hat\lambda(\lambda, A)$ in such a way that the respective gauge transformations satisfy

$$\hat A(A) + \hat\delta_{\hat\lambda} \hat A(A) = \hat A(A + \delta_\lambda A)\,.$$

For rank one and to first order in P, the solution to this differential equation is [15]

$$\begin{aligned}\hat A_\mu(A) &= A_\mu - \frac{1}{2} P^{\rho\sigma} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu})\\ \hat\lambda(\lambda, A) &= \lambda + \frac{1}{4} P^{\mu\nu}\, \partial_\mu\, \lambda A_\nu\,.\end{aligned} \tag{5.3}$$

These transformations can be supersymmetrized as follows. The non-commutative gauge connection and gauge parameter superfields will be now denoted by $\hat V$

and $\hat{\Lambda}$, while we will reserve the notation V and Λ for their ordinary counterparts. The transformations then read,

$$\begin{aligned} \hat{V}(V) &= V + aP^{\mu\nu}\partial_\mu V\nabla_\nu V + (bP^{\alpha\beta}D_\alpha VW_\beta + \text{c.c.}) + (cP^{\alpha\beta}VD_\alpha W_\beta + \text{c.c.})\,, \\ \hat{\Lambda}(\Lambda, V) &= \Lambda + d\bar{D}^2(P^{\alpha\beta}D_\alpha\Lambda D_\beta V)\,, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} P^{\alpha\beta} &= (\sigma_{\mu\nu})^{\alpha\beta}P^{\mu\nu}, \quad (\text{ symmetricin } (\alpha,\beta))\,, \\ W_\alpha &= \bar{D}^2 D_\alpha V\,, \\ \nabla_\nu &= (\sigma_\nu)^{\alpha\dot{\alpha}}[D_\alpha, D_{\dot{\alpha}}]\,. \end{aligned}$$

a, b, c, d are numerical coefficients, which are uniquely fixed in order to reproduce (5.3). We also want to note that $\hat{\Lambda} = \Lambda$ for $D_\alpha\Lambda = 0$, which is required by consistency. Analogously, $\hat{V} = V$ for constant V.

The above results can be easily generalised to non-commutative super-Yang-Mills theories of arbitrary rank.

6. The $\alpha' \mapsto 0$ limit of supersymmetric Born-Infeld action and deformed U(1) gauge theory

In this section we will present the supersymmetric version of the $\alpha' \mapsto 0$ limit of the Born-Infeld action when a B-field is turned on. This action is supposed to describe the deformed version of supersymmetric U(1) gauge theory (in the limit $\alpha' = \mathcal{O}(\epsilon^{1/2}) \mapsto 0$ and slowly varying fields) [15], where the constant field $B^{\mu\nu}$ is related to the Poisson bivector by $B^{\mu\nu} = P^{-1\mu\nu}$ as in (3.4).

Let us first remind the expression obtained in the bosonic case. The lagrangian is given by

$$\mathcal{L}_{BI} = \sqrt{\det(\epsilon^{1/2} + F)} = \sqrt{\epsilon^2 + \frac{\epsilon}{2}F^2 + \frac{1}{16}(F\tilde{F})^2}\,. \tag{6.1}$$

To order ϵ, the lagrangian is readily seen to be

$$\frac{1}{4}|F\tilde{F}| + \epsilon\frac{F^2}{|F\tilde{F}|}\,.$$

To obtain the supersymmetric Born-Infeld action, we will use the following identity [18],

$$\sqrt{X^2 - Y} = X + Y\frac{\sqrt{X^2 - Y} - X}{Y} = X - \frac{Y}{\sqrt{X^2 - Y} + X}\,,$$

where

$$X = \epsilon + \frac{1}{4}F^2, \qquad Y = \frac{1}{16}\Big((F^2)^2 - (F\tilde{F})^2\Big)\,.$$

Denoting by

$$F_\pm = \frac{1}{2}(F \pm \tilde{F})$$

the self dual and anti-self dual combinations of F in an euclidean metric, we also have

$$F_\pm^2 = \frac{1}{2}(F^2 \pm F\tilde{F}), \qquad F_+^2 F_-^2 = \frac{1}{4}\Big((F^2)^2 - (F\tilde{F})^2\Big).$$

We consider a chiral spinor superfield W_α ($\bar{D}_{\dot{\alpha}} W_\alpha = 0$), and the chiral scalar superfield T

$$T = \bar{D}\bar{D}\bar{W}^2 = -\frac{1}{2}F_-^2 + \cdots .$$

We promote X and Y to superfields

$$X = \epsilon - \frac{1}{2}(T + \bar{T}), \quad Y = T\bar{T}.$$

The supersymmetric Born-Infeld action is

$$\mathcal{L}_{\rm SBI} = -\frac{1}{2}\int d^2\theta\, W^2 - \frac{1}{2}\int d^2\bar{\theta}\, \bar{W}^2 - \int d^2\theta d^2\bar{\theta}\, \frac{\bar{W}^2 W^2}{\sqrt{X^2 - Y} + X}. \tag{6.2}$$

One has that

$$\sqrt{X^2 - Y} = \sqrt{\epsilon^2 - \epsilon(T + \bar{T}) + \frac{1}{4}(T - \bar{T})^2}.$$

The order 0 in ϵ in the above expression is the square root of a square, so it should be understood as

$$\sqrt{\frac{1}{4}(T - \bar{T})^2} = \pm\frac{1}{2}(T - \bar{T}) \tag{6.3}$$

depending if

$$\frac{1}{2}(T - \bar{T})|_{\theta=0} = \frac{1}{4}F\tilde{F}$$

is grater or less than zero. So the $\epsilon = 0$ term of $\mathcal{L}_{\rm SBI}$ is

$$\mp\frac{1}{2}\left(\int d^2\,\theta\, W^2 - \int d^2\bar{\theta}\, \bar{W}^2\right) = \frac{1}{4}|F\tilde{F}| + \cdots .$$

For the order ϵ term one gets (in the case with + sign in (6.3))

$$2\epsilon \int d^2\theta\, d^2\bar{\theta}\, \frac{W^2\bar{W}^2}{D^2W^2(D^2W^2 - \bar{D}^2\bar{W}^2)},$$

(for the other case in (6.3) we exchange W by $\bar{W}$ and D by $\bar{D}$).

Finally we get for the $\mathcal{O}(\epsilon)$ in $\mathcal{L}_{\rm SBI}$

$$\begin{aligned}
&\pm\epsilon \int d^2\theta\, d^2\bar{\theta}\, W^2\, \bar{W}^2 \left(\frac{1}{D^2W} + \frac{1}{\bar{D}^2\bar{W}^2}\right)\frac{1}{(D^2W^2 - \bar{D}^2\bar{W}^2)} - \\
&\quad -\epsilon \int d^2\theta d^2\,\bar{\theta}\, \frac{W^2\bar{W}^2}{\left(D^2W^2\,\bar{D}^2\,\bar{W}^2\right)} = \epsilon\left(\frac{F^2}{|F\tilde{F}|} - 1\right) + \cdots .
\end{aligned} \tag{6.4}$$

Note that the last term in (6.4) corresponds to a shift by ϵ in the original Born-Infeld action (6.1),

$$\mathcal{L} = \sqrt{\det(\epsilon^{1/2} + F)} - \epsilon\,.$$

When the B-field is turned on ($F \mapsto F + B$ in the bosonic action), the superfield strength $W_\alpha = \bar{D}\bar{D}D_\alpha V$ (V is the gauge superfield) is shifted into $W_\alpha - L_\alpha$ [46], where L_α is the spinor chiral superfield containing the B-field in its θ-component,

$$L_\alpha = \theta^\beta(\sigma^{\mu\nu}_{\alpha\beta}B_{\mu\nu} + \epsilon_{\alpha\beta}\phi) + \theta^2\chi_\alpha\,,$$

where we used the fact that the combination $W_\alpha - L_\alpha$ is invariant under the superspace gauge transformation

$$\begin{aligned} V &\longmapsto V + U\,,\\ L_\alpha &\longmapsto L_\alpha + \bar{D}^2 D_\alpha U\,, \end{aligned}$$

where U is an arbitrary real scalar superfield. If we want now to compute the supersymmetric Born-Infeld action in the $\epsilon \mapsto 0$ limit with a constant B-field, it is then sufficient to set $\phi = \chi_\alpha = 0$, replace W_α by $W_\alpha - \theta_\beta\sigma^{\mu\nu}_{\alpha\beta}B_{\mu\nu}$ and then use (6.4).

The $\mathcal{O}(\epsilon)$ supersymmetric version of the Born-Infeld bosonic lagrangian,

$$\frac{F^2}{|F\tilde{F}|}$$

has a straightforward generalisation to the case of extended supersymmetry as a full superspace integral[1]. For $N = 2$ theories we have

$$\mathcal{L}_{\mathrm{SBI}}(N=2) = \int d^4\theta\, d^4\bar{\theta}\, \frac{W^2\bar{W}^2}{D^4W^2 - \bar{D}^4\bar{W}^2}\left(\frac{1}{D^4W^2} + \frac{1}{\bar{D}^4\bar{W}^2}\right),$$

where W is the $N = 2$ chiral superfield strength. This is in fact the $\alpha' \mapsto 0$ limit of the $N = 2$ supersymmetric Born-Infeld action [47, 48].

For $N = 4$ we may write an on-shell superspace action [50]

$$\mathcal{L}_{\mathrm{SBI}}(N=4) = \int d^8\theta d^8\bar{\theta}\, \frac{W^{4(0,4,0)}W^{4(0,4,0)}|_{\mathrm{singlet}}}{F_+^2 - F_-^2}\left(\frac{1}{F_+^4F_-^2} + \frac{1}{F_-^4F_+^2}\right),$$

where the $N = 4$ superfield strength $W^{ij} = -W^{ji}$ satisfies the following constraints,

$$\begin{aligned} W^{ij} &= \frac{1}{2}\epsilon^{ijkl}\bar{W}^{kl}\\ \bar{D}_{i\dot{\alpha}}W^{jk} &= \frac{1}{3}\left(\delta_i^j W^{lk} - \delta_i^k \bar{D}_{l\dot{\alpha}}W^{lj}\right),\\ D^i_\alpha W^{jk} &+ D^j_\alpha W^{ik} = 0 \end{aligned}$$

[1] We observe that such generalizations are not unique unless we impose additional requirements on the Born-Infeld action such as electromagnetic duality invariance for its equations of motion [49].

JHEP05(2000)008

and

$$F_+^2 = D^4_{0,2,0} W^{2(0,2,0)}|_{\text{singlet}}\,, \qquad F_-^2 = \bar{D}^4_{0,2,0} W^{2(0,2,0)}|_{\text{singlet}}\,.$$

The indices (a, b, c) refer to the SU(4) Dynkin labels, and "singlet" means a projection on SU(4) invariant combinations.

It is a challenging problem to show that the above actions should reproduce a deformation of the supersymmetric U(1) gauge theory.

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References

[1] H. Weyl, *The theory of groups and quantum mechanics*, Dover, New York 1931.

[2] J. E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Camb. Phil. Soc.* **45** (1949) 99.

[3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization, 1. Deformations of symplectic structures*, *Ann. Phys. (NY)* **111** (1978) 61; *Ann. Phys. (NY)* **111** (1978) 111.

[4] M. Kontsevich, *Deformation quantization of Poisson manifolds, 1*, math.QA/9709040.

[5] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[6] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[8] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034];
D. Brace, B. Morariu and B. Zumino, *Dualities of the matrix model from T-duality of the type-II string*, *Nucl. Phys.* **B 545** (1999) 192 [hep-th/9810099];
D. Brace, B. Morariu and B. Zumino, *T-duality and Ramond-Ramond backgrounds in the matrix model*, *Nucl. Phys.* **B 549** (1999) 181 [hep-th/9811213];
B. Pioline and A. Schwarz, *Morita equivalence and T-duality (or B versus θ)*, *J. High*

Energy Phys. **08** (1999) 021 [hep-th/9908019];
F. Lizzi, *Strings, noncommutative geometry and the size of the target space*, *Int. J. Mod. Phys.* **A 14** (1999) 4501 [hep-th/9902187].

[9] B. Morariu and B. Zumino, *Super Yang-Mills on the noncommutative torus*, hep-th/9807198.

[10] D. Brace and B. Morariu, *A note on the BPS spectrum of the matrix model*, *J. High Energy Phys.* **02** (1999) 004 [hep-th/9810185];
A. Konechny and A. Schwarz, *BPS states on noncommutative tori and duality*, *Nucl. Phys.* **B 550** (1999) 561 [hep-th/9811159];
A. Konechny and A. Schwarz, 1/4-*BPS states on noncommutative tori*, *J. High Energy Phys.* **09** (1999) 030 [hep-th/9907008].

[11] C. Hofman and E. Verlinde, *U-duality of Born-Infeld on the noncommutative two-torus*, *J. High Energy Phys.* **12** (1998) 010 [hep-th/9810116];
C. Hofman and E. Verlinde, *Gauge bundles and Born-Infeld on the noncommutative torus*, *Nucl. Phys.* **B 547** (1999) 157 [hep-th/9810219].

[12] R. D'Auria, S. Ferrara and M.A. Lledó, *On central charges and hamiltonians for 0-brane dynamics*, *Phys. Rev.* **D 60** (1999) 084007 [hep-th/9903089].

[13] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[14] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[15] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[16] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, hep-th/0001203.

[17] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072.

[18] S. Cecotti and S. Ferrara, *Supersymmetric Born-Infeld lagrangians*, *Phys. Lett.* **B 187** (1997) 335.

[19] J. Bagger and A. Galperin, *A new goldstone multiplet for partially broken supersymmetry*, *Phys. Rev.* **D 55** (1997) 1091 [hep-th/9608177].

[20] A. Schwarz, *Noncommutative supergeometry and duality*, hep-th/9912212.

[21] C.-S. Chu and F. Zamora, *Manifest supersymmetry in non-commutative geometry*, *J. High Energy Phys.* **02** (2000) 022 [hep-th/9912153].

[22] D. Minic, *M-theory and deformation quantization*, hep-th/9909022.

[23] A. Jevicki and S. Ramgoolam, *Non-commutative gravity from the AdS/CFT correspondence*, *J. High Energy Phys.* **04** (1999) 032 [hep-th/9902059];
Z. Chang, *Quantum anti-de Sitter space*, hep-th/9904091;
H. Steinacker, *Quantum anti-de Sitter space and sphere at roots of unity*, hep-th/9910037.

[24] R.J. Finkelstein, *Q-gravity*, *Lett. Math. Phys.* **38** (1996) 53.

[25] L. Castellani, *Gauge theories of quantum groups*, *Phys. Lett.* **B 292** (1992) 93 [hep-th/9205103].

[26] F.A. Berezin and M.S. Marinov, *Particle spin dynamics as the Grassmann variant of classical mechanics*, *Ann. Phys. (NY)* **104** (1977) 336.

[27] I. A. Batalin and E. S. Fradkin, *Operator quantization and abelization of dynamical systems subject to first-class constraints*, *Riv. Nuov. Cim.* **9** n.10 (1986) 6; *Operatorial quantization of dynamical systems subject to constraints. A further study of the construction*, *Ann. Inst. Henri Poincaré.* **49**, n. 2 (1988) 145;
M.A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators*, *Fortschr. Phys.* **36** (1988) 33.

[28] D.B. Fairlie and C.K. Zachos, *Infinite dimensional algebras, sine brackets and* $\mathrm{SU}(\infty)$, *Phys. Lett.* **B 224** (1989) 101.

[29] E. S. Fradkin and V. Ya. Linetsky, *Quantization and cocycles on the supertorus and large-N limits for the classical Lie superalgebras*, *Mod. Phys. Lett.* **A 6** (1991) 217.

[30] R. Abraham and J. E. Marsden, *Foundations of mechanics* Addison-Wesley Publishing Company, 1978.

[31] J.L. Martin, *Generalized classical dynamics and the 'classical analogue' of a Fermi oscilator*, *Proc. Roy. Soc.* **A251** (1959) 536.

[32] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained* $N = 2$ *matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469;
E. Ivanov, S. Kalitzin, N. Ai Viet and V. Ogievetsky, *Harmonic superspaces of extended supersymmetry. the calculus of harmonic variables*, *J. Phys.* **18** (1985) 3433.

[33] S. Ferrara and E. Sokatchev, *Short representations of* $\mathrm{SU}(2,2/N)$ *and harmonic superspace analyticity*, hep-th/9912168.

[34] I.Y. Aref'eva, D.M. Belov and A.S. Koshelev, *A note on UV/IR for noncommutative complex scalar field*, hep-th/0001215.

[35] J. Wess and B. Zumino, *Supergauge transformations in four-dimensions*, *Nucl. Phys.* **B 70** (1974) 39;
A lagrangian model invariant under supergauge transformations, *Phys. Lett.* **B 49** (1974) 52.

[36] J. Iliopoulos and B. Zumino, *Broken supergauge symmetry and renormalization*, *Nucl. Phys.* **B 76** (1974) 310.

[37] S. Ferrara, J. Iliopoulos and B. Zumino, *Supergauge invariance and the Gell-Mann–low eigenvalue*, *Nucl. Phys.* **B 77** (1974) 413.

[38] S. Ferrara, J. Wess and B. Zumino, *Supergauge multiplets and superfields*, *Phys. Lett.* **B 51** (1974) 239.

[39] J. Wess and B. Zumino, *Supergauge invariant extension of quantum electrodynamics*, *Nucl. Phys.* **B 78** (1974) 1.

[40] S. Ferrara and B. Zumino, *Supergauge invariant Yang-Mills theories*, *Nucl. Phys.* **B 79** (1974) 413.

[41] I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative R^d, 1. Scalars*, `hep-th/9911098`.

[42] W. Fischler et al., *Evidence for winding states in noncommutative quantum field theory*, *J. High Energy Phys.* **05** (2000) 024 [`hep-th/0002067`].

[43] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, `hep-th/0002075`.

[44] R. Kallosh, *$N = 3$ harmonic superspace*, *Pis'ma ZHETF* **41** (1985) 172, *Sov. Phys. JETP Lett.* **41** (1985) 209.

[45] P.S. Howe and P.C. West, *Operator product expansions in four-dimensional superconformal field theories*, *Phys. Lett.* **B 389** (1996) 273 [`hep-th/9607060`]; *Superconformal invariants and extended supersymmetry*, *Phys. Lett.* **B 400** (1997) 307 [`hep-th/9611075`].

[46] S. Ferrara, M.A. Lledo and A. Zaffaroni, *Born-Infeld corrections to D3 brane action in $AdS_5 \times S^5$ and $N = 4$, $D = 4$ primary superfields*, *Phys. Rev.* **D 58** (1998) 105029 [`hep-th/9805082`].

[47] S.V. Ketov, *A manifestly $N = 2$ supersymmetric Born-Infeld action*, *Mod. Phys. Lett.* **A 14** (1999) 501 [`hep-th/9809121`].

[48] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, `hep-th/9908105`;
M. Roček and A.A. Tseytlin, *Partial breaking of global $D = 4$ supersymmetry, constrained superfields and 3-brane actions*, *Phys. Rev.* **D 59** (1999) 106001 [`hep-th/9811232`].

[49] S.M. Kuzenko and S. Theisen, *Supersymmetric duality rotations*, *J. High Energy Phys.* **03** (2000) 034 [`hep-th/0001068`].

[50] P.S. Howe, K.S. Stelle and P.K. Townsend, *Superactions*, *Nucl. Phys.* **B 191** (1981) 445; *Supercurrents*, *Nucl. Phys.* **B 192** (1981) 332.

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